

AN ALGORITHM TO DETECT FULL IRREDUCIBILITY BY BOUNDING THE VOLUME OF PERIODIC FREE FACTORS

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ABSTRACT. We provide an effective algorithm for determining whether an element ϕ of the outer automorphism group of a free group is fully irreducible. Our method produces a finite list which can be checked for periodic proper free factors.

1. INTRODUCTION

Let F be a finitely generated nonabelian free group of rank at least 2. An outer automorphism ϕ is *reducible* if there exists a free factorization $F = A_1 * \cdots * A_k * B$ such that ϕ permutes the conjugacy classes of the A_i ; else it is *irreducible*. Although irreducible elements have nice properties, e.g., they are known to possess irreducible train-track representatives, irreducibility is not preserved under iteration. Thus one often considers elements that are *irreducible with irreducible powers (iwip)*, or *fully irreducible*. These are precisely the outer automorphisms ϕ for which there does not exist a proper free factor $A < F$ whose conjugacy class $[A]$ satisfies $\phi^p([A]) = [A]$ for any $p > 0$. If $\phi^p([A]) = [A]$ for some proper free factor $A < F$ and for some $p > 0$, we say $[A]$ is ϕ -periodic; and, to avoid sometimes cumbersome language, also that the free factor A is ϕ -periodic. Fully irreducible elements are considered analogous to pseudo-Anosov mapping classes of hyperbolic surfaces. As such, they play an important role in the geometry and dynamics of the outer automorphism group $\text{Out}(F)$ of F .

Although considered in some sense a “generic” property in $\text{Out}(F)$, full irreducibility is not generally easy to detect. Kapovich [12] gave an algorithm for determining whether a given $\phi \in \text{Out}(F)$ is fully irreducible, inspired by Pfaff’s criterion for full irreducibility in [16]. At points in his algorithm, two processes run simultaneously, and although it is known that one of these must terminate, it is not *a priori* known

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which will; it thus seems unclear that the complexity of Kapovich’s algorithm can be found without running the algorithm itself.

For mapping class groups, there exist algorithms for determining whether or not a given mapping class is pseudo-Anosov due to Bestvina–Handel and Chen–Hamidi–Tehrani [4, 5]. The algorithm of Chen–Hamidi–Tehrani is exponential in terms of the number of generators needed to write the given mapping class. More recently, Koberda and the second author [13] provided an elementary algorithm for determining whether or not a given mapping class is pseudo-Anosov, using a method of “list and check.” They show that if a mapping class f is reducible, i.e., has an invariant multicurve, then the curves in its reduction system have length bounded by an exponential function in terms of the number of generators needed to write f . Therefore, given a mapping class f , a **list** is produced of all multicurves whose curves are sufficiently short. The action of f is then **checked** on these finitely many multicurves. If f fixes a multicurve from the list, it is reducible; otherwise, it is necessarily pseudo-Anosov.

In this article, we provide, in essence, a method of “list and check” for elements of $\text{Out}(F)$, akin to that of Koberda and the second author. That is, we provide an algorithm which, given an element ϕ , expressed as a product of generators from a finite generating set of $\text{Out}(F)$, produces a finite list of free factors and checks each for ϕ -periodicity. The algorithm effectively determines whether or not the given element ϕ is fully irreducible. By *effective*, we understand that there is a computable function which bounds the number of steps in terms of the size of the input and that does not utilize the algorithm. In particular, we avoid the use of dual infinite processes, one of which must terminate.

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2. STATEMENT OF RESULTS

By $\text{rk}(F)$ we denote the rank of the free group F . Let $\xi(F) = 3\text{rk}(F) - 3$. This is the maximum number of edges in a finite graph with fundamental group F and without degree one or two vertices. This is also the maximum number of isotopy classes of disjoint essential (not bounding a ball) spheres in the double of the handlebody of genus $\text{rk}(F)$. An element $\phi \in \text{Out}(F)$ that is not fully irreducible is *cyclically reducible* if there exists a ϕ -periodic rank 1 free factor; else it is *noncyclically reducible*.

Our algorithm to determine full irreducibility of an element $\phi \in \text{Out}(F)$ consists of two effective processes. Process I determines (in the absence of an obvious reduction) if ϕ is cyclically reducible. As we shall see in Section 3, this will exploit algorithms which are already well-known. Our main contribution to the algorithm is in process II. For this we construct a finite list of conjugacy classes of proper free factors that contains a ϕ -periodic free factor if ϕ is noncyclically reducible. The length of this list is controlled by the word length of ϕ ; this is the content of Theorem 1. A systematic check of the list then determines whether or not ϕ is fully irreducible.

To state our main theorem, we start by fixing a basis \mathcal{X} for the free group F , and let $T = T_{\mathcal{X}}$ denote the Cayley graph for F with respect to \mathcal{X} . Given a subgroup $A \leq F$, the *volume* $\|A\|_{\mathcal{X}}$ of A is the number of edges in the *Stallings core* of the graph T/F . Recall that the Stallings core is the graph T_A/A , where T_A is the minimal subtree of T with respect to the action of A ; or, equivalently, the Stallings core is the smallest subgraph of the cover of T/F associated to A that contains every embedded cycle (see [17] for details). Note that the volume function $\|\cdot\|_{\mathcal{X}}$ is constant on conjugacy classes of subgroups. The quantity $\|A\|_{\mathcal{X}}$ gives some measure of the complexity of the subgroup A in terms of the basis \mathcal{X} . For instance, if $A = \langle a \rangle$ is a cyclic subgroup, the volume $\|\langle a \rangle\|_{\mathcal{X}}$ is the cyclic length of the element a as a word in the basis \mathcal{X} .

Now fix a finite generating set \mathcal{S} for $\text{Out}(F)$. Denote by $|\phi|_{\mathcal{S}}$ the word length of $\phi \in \text{Out}(F)$ with respect to \mathcal{S} . Our main theorem describes a relation between the word length of a noncyclically reducible element of $\text{Out}(F)$ and the volume of one of its periodic free factors.

Theorem 1. *There is a computable constant $C = C(\mathcal{X}, \mathcal{S})$ such that if $\phi \in \text{Out}(F)$ is noncyclically reducible, then there is a ϕ -periodic proper free factor A such that $\|A\|_{\mathcal{X}} \leq C^{|\phi|_{\mathcal{S}}}$.*

As there are a finite number of conjugacy classes of free factors A of F for which $\|A\|_{\mathcal{X}}$ is bounded, the theorem provides a bound for the size of a list of conjugacy classes of free factors that can be used to conclusively determine whether or not an element $\phi \in \text{Out}(F)$ of length $|\phi|_{\mathcal{S}}$ is fully irreducible, if ϕ is not cyclically reducible.

To prove Theorem 1, we utilize a notion of *intersection number* $i(S, T)$ defined between a pair of trees S and T equipped with an isometric action by F , as defined by Guirardel [8]. Horbez [11] related the intersection number $i(T, T\phi)$ to the word length of $\phi \in \text{Out}(F)$ (see Theorem 5 and Corollary 7 in Section 4). We thus need only bound

the volume of a ϕ -periodic proper free factor by $i(T, T\phi)$ (Section 6, Proposition 10).

Before embarking on the details of the proof of Theorem 1, we will first describe the procedure used in our algorithm for detecting fully irreducible elements of $\text{Out}(F)$. This is contained in the next section, where we establish:

Theorem 2. *There exists an effective algorithm for determining if an outer automorphism is fully irreducible.*

3. LIST AND CHECK ALGORITHM

The input of our algorithm is an element $\phi_0 \in \text{Out}(F)$. Recall that $\phi_0 \in \text{Out}(F)$ is not fully irreducible if there exists a periodic proper free factor, and note that the periodic free factors of ϕ_0 are exactly the periodic free factors of each of its powers. Feighn and Handel [6] showed that there is a power Q , depending on the rank of F but not on the element ϕ_0 , so that any periodic free factor of ϕ_0^Q is in fact invariant (see also [9]). This is analogous to the fact that the mapping class group has a finite index subgroup all of whose elements are *pure*; i.e., any invariant multicurve is curve-wise fixed. As a preliminary step to our algorithm, we replace the element ϕ_0 by $\phi = \phi_0^Q$, so that henceforth we need only look for invariant free factors. Note that ϕ is irreducible if and only if it is fully irreducible if and only if ϕ_0 is fully irreducible.

Process I. To begin process I, we apply an algorithm due to Bestvina and Handel [4] which finds a *relative train track*¹ representative $f: \Gamma \rightarrow \Gamma$ of ϕ . At its conclusion, if Γ has a nontrivial f -invariant subgraph, then ϕ fixes a proper free factor and is therefore reducible. Otherwise, the algorithm gives us an honest train track map representing ϕ . Recall that Bestvina and Handel [4] proved that the fixed subgroup of an automorphism whose outer class is irreducible is at most rank 1. Thus we next want to check for loops homotopically fixed by f , which correspond to a fixed conjugacy classes of ϕ , and then see whether their corresponding elements generate a higher rank subgroup of F .

For this, we make use of an algorithm described by Turner [18] for determining the (finite number of) indivisible Nielsen paths, i.e., minimal paths fixed up to homotopy, of a train track map. The indivisible

¹Loosely speaking, a relative train track representative is akin to a Jordan form for a linear transformation; we will not make use of any properties of relative train track representatives and refer the reader to the references for details.

Nielsen paths define a graph C_f and a map $C_f \rightarrow \Gamma$ that is (homotopically) invariant under f . If some component of C_f has rank greater than 1, then there is an automorphism in the outer class of ϕ which has a fixed subgroup of rank greater than 1 and hence ϕ is reducible and we stop. Else, for each rank 1 component, Whitehead's algorithm [20] is applied to determine whether the corresponding element is primitive. If some component is, then ϕ is cyclically reducible. This marks the end of process I. At this point, we have determined if ϕ is reducible or if ϕ is *not* cyclically reducible, i.e., either noncyclically reducible or fully irreducible. In the former case we stop, else we continue onto process II.

Process II. Theorem 1 gives an upper bound $V = C^{|\phi|_s}$ on the volume of the smallest ϕ -invariant free factor, if ϕ is noncyclically reducible. (Recall, we have replaced our original input ϕ_0 by $\phi = \phi_0^Q$ for which invariance and periodicity are the same.) There are a finite number of conjugacy classes of subgroups H with volume less than this bound, and these can be systematically listed, since they correspond to core graphs made from at most V edges, where each edge is oriented and labeled by an element of \mathcal{X} . For a gross overestimate of the number of these, one has $V \cdot (2\text{rk}(F))^V \cdot B_{2V}$, where B_n , known as the *n*th *Bell number*, counts the number of partitions of n objects. In our case this is equivalent to the number of ways one can glue the $2V$ endpoints of V edges to obtain a graph. In particular, the number of conjugacy classes is less than $V(8V^2\text{rk}(F))^V$ as $B_{2V} \leq (2V)^{2V}$. Whitehead's algorithm is then used to eliminate conjugacy classes which are not free factors. We obtain a **list** of conjugacy classes of free factors that are **checked** (using, say, Stallings's graph pull backs [17]) one-by-one for ϕ -invariance. This process, and hence the algorithm, stops once either an invariant free factor is identified, concluding with ϕ reducible, or once every item on the list is checked and found not to be invariant, determining that ϕ is fully irreducible.

This completes the proof of Theorem 2, with the assumption of Theorem 1. Now we proceed with the proof of Theorem 1.

4. OUTER SPACE AND THE GUIARDEL CORE

For mapping class groups, the intersection number between curves on the surface is in various contexts useful in comparison to distances in, for instance, Teichmüller space or the complex of curves. Similar methods have been emerging for $\text{Out}(F)$ and its associated spaces. Culler and Vogtmann's *outer space* is the space cv consisting of marked metric simplicial trees T with a minimal, simplicial, free F -action, up

to isometry which commutes with the action. The action of $\text{Out}(F)$ on cv is defined by pre-composing the free group action with the outer automorphism; this action is therefore on the right. In some contexts, it is convenient to consider the projectivized outer space CV in which the sum of the lengths of the edges of the quotient T/F is 1. The space CV is treated as the analogue for $\text{Out}(F)$ of Teichmüller space; we refer the reader to Vogtmann's survey [19] for a more detailed description of CV .

The utility of the intersection number between curves on a surface is carried over to free groups via the so-called *Guirardel core* $\mathcal{C}(S \times T)$: a certain closed, F -invariant (with the diagonal action) cellular subset of the product $S \times T$ of trees in $S, T \in cv$. The *intersection number* $i(S, T)$ is the covolume of $\mathcal{C}(S \times T)$; that is, the sum of the areas of the 2-cells in $\mathcal{C}(S \times T)/F$. For our purposes, we will not need the full definition of the core, for which we refer the reader to [8, 11]. Rather, Behrstock, Bestvina and the first author [2] gave a simple criteria for when two edges $s \in \mathcal{E}(S)$, $t \in \mathcal{E}(T)$ determine a square $s \times t$ in the core $\mathcal{C}(S \times T)$, and it is this that we require in the sequel.

For a tree $T \in cv$, we let ∂T denote its boundary; that is, equivalence classes of geodesic rays where two rays are equivalent if their images lie in a bounded neighborhood of one another. An oriented edge $t \in \mathcal{E}(T)$ determines a subset $\text{Cyl}_T^1(t)^+$, the *one-sided cylinder*, which consists of equivalence classes of geodesics that contain a representative whose image contains t with the correct orientation. The complement of $\text{Cyl}_T^1(t)^+$ in ∂T will be denoted by $\text{Cyl}_T^1(t)^-$, where clearly $\text{Cyl}_T^1(t)^- = \text{Cyl}_T^1(\bar{t})^+$. We will typically not bother with specifying an orientation as we will consider both one-sided cylinders simultaneously.

Lemma 3 (Lemma 2.3 [2]). *Let $S, T \in cv$ and let $\partial: \partial S \rightarrow \partial T$ denote the canonical F -equivariant homeomorphism. Given two edges $s \in \mathcal{E}(S)$ and $t \in \mathcal{E}(T)$, the rectangle $s \times t$ is in the core $\mathcal{C}(S \times T)$ if and only if each of the four subsets $\partial(\text{Cyl}^1(s)^{(\pm)}) \cap \text{Cyl}^1(t)^{(\pm)}$ is nonempty.*

Let $S, T \in cv$ and $t \in \mathcal{E}(T)$. The *slice* of the core $\mathcal{C}(S \times T)$ above t is the set:

$$\mathcal{C}_t = \{s \in \mathcal{E}(S) \mid s \times t \subset \mathcal{C}(S \times T)\}.$$

Similarly define the slice $\mathcal{C}_s = \{t \in \mathcal{E}(T) \mid s \times t \subset \mathcal{C}(S \times T)\}$ for $s \in \mathcal{E}(S)$. A simple application of Lemma 3 can be used to describe the slice.

Lemma 4 (Lemma 3.7 [2]). *Let $S, T \in cv$ and suppose $f: S \rightarrow T$ is a linear F -equivariant map. Given an edge $t \subset T$ and a point x in the*

interior of t , the slice $\mathcal{C}_t \subset S$ of the core $\mathcal{C}(S \times T)$ is contained in the subtree spanned by $f^{-1}(x)$.

As F acts freely on the edges of T , for any point x that is in the interior of t , the subtree $\mathcal{C}_t \times \{x\}$ embeds in the quotient $\mathcal{C}(S \times T)/F$. Similarly, for a point x in the interior of s , the subtree $\{x\} \times \mathcal{C}_s$ embeds in the quotient. Therefore, the intersection number $i(S, T)$ can be expressed as:

$$i(S, T) = \sum_{e \in \mathcal{E}(T/F)} \ell_T(\tilde{e}) \text{vol}(\mathcal{C}_{\tilde{e}}) = \sum_{e \in \mathcal{E}(S/F)} \ell_S(\tilde{e}) \text{vol}(\mathcal{C}_{\tilde{e}}). \quad (1)$$

where by \tilde{e} we denote any lift of the edge e to T or S respectively and by $\text{vol}(\cdot)$ we denote the sum of the lengths of the edges in the respective slice.

Intersection numbers for free groups can also be interpreted as the geometric intersection between sphere systems in the doubled handlebody. This connection is recalled in the proof of Lemma 9, where it will be used.

There is a natural $\text{Out}(F)$ -invariant “metric” on CV with respect to the action of $\text{Out}(F)$. The *Lipschitz (asymmetric) metric* $d(\cdot, \cdot)$ measures the optimal maximal stretch along edges of an F -equivariant map between two trees. That is, $d(S, T) = \log \text{Lip}(S, T)$ where $\text{Lip}(S, T)$ is the infimum of the Lipschitz constants over all F -equivariant maps $S \rightarrow T$. We refer the reader to [1, 7] for complete details, but we remark here that the infimum is obtained by some map, and such a map is called *optimal*.

As mentioned in Section 2, Horbez [11] has recently given, for two trees in CV , the following relation between their Guirardel intersection number and their distance in CV .

Theorem 5 (Horbez [11]). *For all $\epsilon > 0$, there exist a computable $C_0 = C_0(\text{rk}(F), \epsilon)$ such that for all trees S and T in the ϵ -thick part of CV , we have*

$$\frac{1}{C_0} \log(i(S, T)) - C_0 \leq d(S, T).$$

Here the ϵ -thick part of outer space refers to the subset of trees for which all elements of F have translation length at least ϵ . We remark that our definition of $i(S, T)$ sums the areas of 2-cells in the Guirardel core, whereas Horbez in [11] defines $i(S, T)$ as a tally of the number of 2-cells in the core. Because the second definition results in a strictly larger value for S, T in projectivized outer space, this difference affects neither our statement of his theorem above nor our use of it. In our application we will have $\epsilon = 1/\text{rk}(F)$.

Let $\overline{T}_\mathcal{X} \in CV$ be the Cayley tree for the basis \mathcal{X} with edges scaled to have equal length $1/\text{rk}(F)$. We have, by the triangle inequality:

Lemma 6. *There exists a computable $C_1 = C_1(\mathcal{X}, \mathcal{S})$ such that for all $\phi \in \text{Out}(F)$,*

$$d(\overline{T}_\mathcal{X}, \overline{T}_\mathcal{X}\phi) \leq C_1|\phi|_\mathcal{S}.$$

□

It now follows easily from Theorem 5 and Lemma 6 that:

Corollary 7. *There exists a computable $C_2 = C_2(\mathcal{X}, \mathcal{S})$ such that for all $\phi \in \text{Out}(F)$,*

$$i(\overline{T}_\mathcal{X}, \overline{T}_\mathcal{X}\phi) \leq C_2^{|\phi|_\mathcal{S}}.$$

□

5. INTERSECTION NUMBERS, SUBGROUPS, AND VOLUMES

Given trees $S, T \in cv$ and a nontrivial finitely generated subgroup $A \leq F$, there exist nonempty subtrees $S_A \subset S$, $T_A \subset T$, on each of which A acts minimally. We can thus consider the Guirardel core $\mathcal{C}(S_A \times T_A)$ for these minimal subtrees with respect to the action of A . We might hope that, if A is a free factor of F , then $\mathcal{C}(S_A \times T_A)$ embeds into $\mathcal{C}(S \times T)$, so that $i(S_A, T_A)$ is always dominated by $i(S, T)$. Unfortunately this appears to be too much to expect, but we do achieve:

Proposition 8. *Let A be a noncyclic finitely generated subgroup of F and $\epsilon > 0$. Suppose that $S, T \in cv$ are such that the length of the shortest edge in T is at least ϵ and let $S_A \subset S$, $T_A \subset T$ be the minimal subtrees with respect to A . Then:*

$$i(S_A, T_A) \leq \xi(A) \cdot \frac{\text{vol}(S/F) \text{Lip}(S, T)^2}{\epsilon} \cdot i(S, T). \quad (2)$$

Proof. By Equation (1), we have:

$$\begin{aligned} i(S, T) &= \sum_{e \in \mathcal{E}(T/F)} \ell_T(\tilde{e}) \text{vol}(\mathcal{C}_{\tilde{e}}) \\ i(S_A, T_A) &= \sum_{e \in \mathcal{E}(T_A/A)} \ell_{T_A}(\tilde{e}) \text{vol}(\mathcal{A}_{\tilde{e}}) \end{aligned} \quad (3)$$

where $\mathcal{C}_{\tilde{e}} \subset S$ and $\mathcal{A}_{\tilde{e}} \subset S_A \subset S$ are the slices in the respective cores. We denote by $\partial: \partial S \rightarrow \partial T$, the canonical F -equivariant homeomorphism, and by $\partial_A: \partial S_A \rightarrow \partial T_A$, the canonical A -equivariant homeomorphism. Observe that $\partial|_{\partial S_A} = \partial_A$.

First, we claim that for each edge $\tilde{e} \subset T_A \subset T$ we have that $\mathcal{A}_{\tilde{e}} \subseteq \mathcal{C}_{\tilde{e}}$. Indeed, let s be an edge in $\mathcal{A}_{\tilde{e}}$. By Lemma 3, each of the four sets

$\partial_A(Cyl_{S_A}^1(s)^{(\pm)}) \cap Cyl_{T_A}^1(\tilde{e})^{(\pm)}$ is non-empty. As $\partial_A(Cyl_{S_A}^1(s)^{(\pm)}) = \partial(Cyl_{S_A}^1(s)^{(\pm)}) \subset \partial(Cyl_S^1(s)^{(\pm)})$ and $Cyl_{T_A}^1(\tilde{e})^{(\pm)} \subset Cyl_T^1(\tilde{e})^{(\pm)}$, each of the four sets $\partial(Cyl_S^1(s)^{(\pm)}) \cap Cyl_T^1(\tilde{e})^{(\pm)}$ is nonempty. Hence s is an edge in $\mathcal{C}_{\tilde{e}}$.

By a *natural edge* of T_A we mean an edge path $\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n$ that is a connected component of $T_A - \mathcal{V}_{\geq 3}(T_A)$, where $\mathcal{V}_{\geq 3}(T_A)$ is the collection of vertices of degree at least three. A natural edge in T_A/A is the image of a natural edge in T_A ; the set of all natural edges is denoted $\mathcal{E}_N(T_A/A)$.

Suppose that \tilde{e} is a natural edge of T_A consisting of the edge path $\tilde{e}_1, \tilde{e}_2, \dots, \tilde{e}_n$. Then since $Cyl_{T_A}^1(\tilde{e}_i)^{(\pm)} = Cyl_{T_A}^1(\tilde{e}_j)^{(\pm)}$ for all i, j , from Lemma 3 we see that $\mathcal{A}_{\tilde{e}_i} = \mathcal{A}_{\tilde{e}_j}$. Therefore, we are justified in writing $\mathcal{A}_{\tilde{e}}$ to denote any of the slices $\mathcal{A}_{\tilde{e}_i}$. Applying the observation that $\mathcal{A}_{\tilde{e}_i} \subseteq \mathcal{C}_{\tilde{e}_i}$ we have that:

$$\mathcal{A}_{\tilde{e}} \subseteq \bigcap_{i=1}^n \mathcal{C}_{\tilde{e}_i} \quad (4)$$

We claim that, if $\ell_{T_A}(\tilde{e}) > \text{vol}(S/F) \text{Lip}(S, T)^2$, then $\mathcal{A}_{\tilde{e}} = \emptyset$. Indeed, we will show that $\mathcal{C}_{\tilde{e}_1} \cap \mathcal{C}_{\tilde{e}_n} = \emptyset$ in this case. Let $f: S \rightarrow T$ be an optimal map, so that $\text{Lip}(f) = \text{Lip}(S, T)$. Choose interior points $x_1 \in \tilde{e}_1$ and $x_n \in \tilde{e}_n$ such that $d_T(x_1, x_n)$ is arbitrarily close to $\ell_{T_A}(\tilde{e})$. Any two points in $f^{-1}(x_1)$ have distance at most $\text{vol}(S/F) \text{Lip}(f)$ [3, Lemma 3.1]; the same is true for any two points in $f^{-1}(x_n)$. Additionally, the distance between any point in $f^{-1}(x_1)$ and any point in $f^{-1}(x_n)$ is at least $\ell_{T_A}(\tilde{e}) / \text{Lip}(f) > \text{vol}(S/F) \text{Lip}(f)$. Therefore, no point in S can be in the span of both $f^{-1}(x_1)$ and $f^{-1}(x_n)$. By Lemma 4, the slice $\mathcal{C}_{\tilde{e}_1}$ is contained in the subtree spanned by $f^{-1}(x_1)$, and the slice $\mathcal{C}_{\tilde{e}_n}$ is contained in the subtree spanned by $f^{-1}(x_n)$. This shows that $\mathcal{C}_{\tilde{e}_1} \cap \mathcal{C}_{\tilde{e}_n} = \emptyset$, as claimed.

Now rewriting (3) we get:

$$\begin{aligned} i(S_A, T_A) &= \sum_{e \in \mathcal{E}(T_A/A)} \ell_{T_A}(\tilde{e}) \text{vol}(\mathcal{A}_{\tilde{e}}) \\ &= \sum_{e \in \mathcal{E}_N(T_A/A)} \ell_{T_A}(\tilde{e}) \text{vol}(\mathcal{A}_{\tilde{e}}) \\ &\leq \sum_{e \in \mathcal{E}_N(T_A/A)} \text{vol}(S/F) \text{Lip}(S, T)^2 \text{vol}(\mathcal{A}_{\tilde{e}}) \\ &= \frac{\text{vol}(S/F) \text{Lip}(S, T)^2}{\epsilon} \sum_{e \in \mathcal{E}_N(T_A/A)} \epsilon \text{vol}(\mathcal{A}_{\tilde{e}}) \end{aligned}$$

Since $\epsilon \operatorname{vol}(\mathcal{A}_{\tilde{e}}) \leq i(S, T)$ for every natural edge $\tilde{e} \in \mathcal{E}_N(T_A/A)$ and as the number of natural edges in T_A/A is at most $\xi(A)$, Equation (2) holds. \square

In order to eventually relate intersection number to the volume of an invariant free factor, we find an effective lower bound for intersection under a bounded iterate of a fully irreducible automorphism.

Lemma 9. *Suppose F is free of rank at least 2. Let ϕ be a fully irreducible element of $\operatorname{Out}(F)$ and consider a tree $T \in cv$ with edge lengths greater than 1. Then for some $1 \leq P \leq \xi(F)$:*

$$\operatorname{vol}(T/F) \leq \xi(F) \cdot i(T, T\phi^P).$$

Proof. The lemma will be proved once we establish that, for some $1 \leq P \leq \xi(F)$, the slice of the core $\mathcal{C}(T \times T\phi^P)$ above the longest edge of T contains at least one edge of $T\phi^P$ and hence has volume at least 1. This is because, if the longest edge is \tilde{e} , by Equation (1) we would have:

$$\operatorname{vol}(T/F) \leq \xi(F) \cdot \ell_T(\tilde{e}) \leq \xi(F) \cdot \ell_T(\tilde{e}) \cdot \operatorname{vol}(\mathcal{C}_{\tilde{e}}) \leq \xi(F) \cdot i(T, T\phi^P).$$

To prove the claim above, we compute the intersection number using *sphere systems* in the doubled handlebody with fundamental group F . Briefly, let M be the connect sum of as many copies of $S^1 \times S^2$ as the rank of F . By \mathbb{S} we denote the simplicial complex whose n -simplices correspond to $n+1$ isotopy classes of disjoint essential spheres in M , and by \mathbb{S}^∞ we denote the subcomplex of \mathbb{S} consisting of simplices where the complement of the corresponding sphere system in M has a nonsimply-connected component. By work of Laudenbach [14, 15], there is a well-defined simplicial action of $\operatorname{Out}(F)$ on \mathbb{S} that leaves \mathbb{S}^∞ invariant. In this action, fully irreducible elements of $\operatorname{Out}(F)$ act on \mathbb{S} without periodic orbits. Hatcher [10] established an $\operatorname{Out}(F)$ -equivariant isomorphism between projectivized outer space CV and $\mathbb{S} - \mathbb{S}^\infty$. Under this isomorphism, edges of a marked graph T/F correspond bijectively to spheres in some sphere system.

Horbez details the correspondence between geometric intersection of the sphere systems and the volume of the Guirardel core [11]. In particular he shows that, if T_0, T_1 are trees in CV , then for the corresponding sphere systems $\Sigma_0, \Sigma_1 \in \mathbb{S} - \mathbb{S}^\infty$, we have $i(T_0, T_1) = i(\Sigma_0, \Sigma_1)$, where the latter counts the minimal number of circles common to each sphere system, weighted appropriately. While not stated explicitly in [11], it can be verified that each circle of intersection occurring on a given component $\sigma_0 \in \Sigma_0$ corresponds to an edge in the slice of the core $\mathcal{C}(T_0 \times T_1)$ above an edge in T_0 corresponding to the lift of the edge in T_0/F dual to

σ_0 . For our considerations, the weights on the spheres will not matter as we are only concerned with showing that some slice is nonempty, i.e., that the corresponding sphere has nontrivial intersection with another sphere.

Now, given $T \in cv$, we scale its edges equally by $1/\text{vol}(T/F)$ to get a point $\bar{T} \in CV$. The slice over the longest edge of T in $\mathcal{C}(T \times T\phi^P)$ is non-empty if and only if the slice over the longest edge of \bar{T} in $\mathcal{C}(\bar{T} \times \bar{T}\phi^P)$ is non-empty. Suppose Σ is the sphere system dual to \bar{T} and that $\sigma \in \Sigma$ is dual to the longest edge of \bar{T} . As the maximum number of isotopy classes of disjoint essential spheres in M is $\xi(F)$ and as ϕ is fully irreducible, at least two of the spheres $\sigma, \phi(\sigma), \dots, \phi^{\xi(F)}(\sigma)$ have essential intersection, so in particular σ essentially intersects $\phi^P(\sigma)$ for some $1 \leq P \leq \xi(F)$. By the foregoing discussion, this means the slice above the longest edge in T/F contains at least one edge, proving the lemma. \square

We apply the previous two results to prove:

Proposition 10. *Let $T = T_{\mathcal{X}}$ be the Cayley graph with respect to the basis \mathcal{X} , with all edges of unit length. If $\phi \in \text{Out}(F)$ acts fully irreducibly on a proper free factor A of rank at least 2, then for some $1 \leq P \leq \xi(F)$:*

$$\|A\|_{\mathcal{X}} \leq \xi(F)^3 \cdot \text{Lip}(T, T\phi^P)^2 \cdot i(T, T\phi^P).$$

Proof. The minimal tree $T_A \subset T$ of A has natural edge-lengths at least 1, and as such can be thought of as an element of $cv(A)$, the unprojectivized outer space for A . We can apply Lemma 9 to T_A with its free A -action to obtain $P \leq \xi(A)$ for which

$$\|A\|_{\mathcal{X}} = \text{vol}(T_A/A) \leq \xi(A) \cdot i(T_A, T_A\phi^P).$$

The conclusion follows from applying Proposition 8, using $\text{vol}(T/F) \leq \xi(F)$ and $\epsilon = 1$, while noting that $\xi(A) \leq \xi(F)$. \square

6. PROOF OF THEOREM 1

In this section we prove the key new result for our algorithm, which is Theorem 1. We wish to show that if $\phi \in \text{Out}(F)$ is noncyclically reducible there is a ϕ -periodic free factor whose volume is bounded above by an exponential function in terms of the word length $|\phi|_S$. Since ϕ is noncyclically reducible, there is a ϕ^Q -invariant free factor A for which $1 < \text{rk}(A) < \text{rk}(F)$, where $Q = Q(\text{rk}(F))$ is the constant power mentioned at the beginning of Section 3. We can assume that $\phi^Q|_A$ is fully irreducible.

Now let us complete the proof of Theorem 1. Recall that $\overline{T} = \overline{T}_{\mathcal{X}}$ is obtained from T by scaling each edge by $1/\text{rk}(F)$. We can relate intersection numbers now as follows:

$$i(\overline{T}, \overline{T}\phi^p) = \frac{1}{\text{rk}(F)^2} i(T, T\phi^p)$$

for any p . Combining with Proposition 10 and Corollary 7, we have some $1 \leq P \leq \xi(F)$ for which

$$\begin{aligned} \|A\|_{\mathcal{X}} &\leq \xi(F)^5 \cdot \text{Lip}(T, T\phi^{Q^P})^2 \cdot i(\overline{T}, \overline{T}\phi^{Q^P}) \\ &\leq \xi(F)^5 \cdot \text{Lip}(T, T\phi^{Q^P})^2 \cdot C_2^{|\phi^{Q^P}|_{\mathcal{S}}}. \end{aligned}$$

Let $C_3 = \max\{\text{Lip}(T, T\psi) \mid \psi \in \mathcal{S}\}$. This constant is computable and only depends on \mathcal{X} and \mathcal{S} . Because the product of Lipschitz constants of two functions provides an upper bound for the Lipschitz constant of the composition of those functions, we have $\text{Lip}(T, T\phi^{Q^P}) \leq C_3^{|\phi^{Q^P}|_{\mathcal{S}}}$.

The proof of Theorem 1 is complete with:

$$\begin{aligned} \|A\|_{\mathcal{X}} &\leq \xi(F)^5 \cdot (C_3^{|\phi^{Q^P}|_{\mathcal{S}}})^2 \cdot C_2^{|\phi^{Q^P}|_{\mathcal{S}}} \\ &\leq C^{|\phi|_{\mathcal{S}}}, \end{aligned}$$

where $C = \xi(F)^5 \cdot (C_3^2 \cdot C_2)^{Q\xi(F)}$ depends only on \mathcal{X} and \mathcal{S} .

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